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LETTER TO THE EDITOR

Bosonic Fock representations of the affine-Virasoro algebra

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Abstract. Bosonic Fock representations of the Virasoro algebra and the affine Lie algebras attached to classical Lie algebras with non-zero centres are explicitly presented. Based on these, Fock representations of the affine-Virasoro algebra are constructed. An automorphism of the affine-Virasoro algebra is also given.

In recent years, the Virasoro algebra and affine Lie algebras have attracted more and more attention from both mathematicians and physicists because of their increasing importance in such domains of theoretical physics as soliton theory, conformal field theory and quantum inverse scattering theory [1–3]. At first sight, these two types of infinite dimensional Lie algebras seem quite different, but this is an illusion. As a matter of fact, the Virasoro algebra is a non-trivial central extension of the algebra of vector fields on the circle S^1 and an affine Lie algebra is nothing else but a central extension of a loop algebra, which can be regarded as the Lie algebra of a loop group (mappings from S^1 to a compact Lie group) [4]. So the Virasoro algebra has a natural action on an affine Lie algebra. This close relationship strongly suggests that they be considered simultaneously, i.e. as one algebraic structure, and hence has led to the definition of the so-called affine-Virasoro algebra [5]. As for the bosonic Fock representation of the affine-Virasoro algebra, the famous Sugawara construction provides the first example [5, 6], and as far as we know it is also the unique type of bosonic Fock representation presently known. On the other hand, there have been some free fermion realizations of the affine-Virasoro algebra [7–9]. In this letter we will rigorously establish a new kind of representation of the affine-Virasoro algebra on bosonic Fock spaces, which is much simpler than that given by the Sugawara construction.

Suppose g is an arbitrary Lie algebra over the complex number field $\mathbb{C}$, then the loop algebra $L(g)$ attached to g is defined by

$$L(g) = g \otimes \mathbb{C}[t, t^{-1}].$$

If a symmetric bilinear form $\langle . \rangle$ on g satisfying the condition

$$\langle A, [B, C]\rangle = \langle [A, B], C\rangle \qquad \forall A, B, C \in g$$

is given, $L(g)$ can be extended to the affine Lie algebra $\hat{g}$:

$$\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus c$$

with the commutation relation

$$\begin{aligned} &[X(m), Y(n)] = [X, Y](m+n) + m\langle X, Y\rangle \delta_{m,n} c \qquad \forall X, Y \in g \\ &[c, X(m)] = 0 \end{aligned} \tag{1}$$

where $X(m) = X \otimes t^m$.

Now denote by δ the Lie algebra $\mathbb{C}[t, t^{-1}](\mathrm{d}/\mathrm{d}t)$ of polynomial vector field on the circle. Obviously, the elements $d_m = -t^{m+1}(\mathrm{d}/\mathrm{d}t)(m \in \mathbb{Z})$ constitute a basis of $\mathbb{C}[t, t^{-1}](\mathrm{d}/\mathrm{d}t)$. The so-called Virasoro algebra is, by definition, a non-trivial central extension of $\mathbb{C}[t, t^{-1}](\mathrm{d}/\mathrm{d}t)$. Namely, the Virasoro algebra Vir is a Lie algebra with the basis $\{d_m, m \in \mathbb{Z}, \tilde{c}\}$ and the following commutation relations

$$\begin{aligned} &[d_m, d_n] = (m-n)d_{m+n} + \delta_{m,-n}\tilde{c}(m^3 - m)/2 \\ &[d_m, \tilde{c}] = 0. \end{aligned} \tag{2}$$

As is easily seen, the Lie algebra $\mathbb{C}[t, t^{-1}]\frac{\mathrm{d}}{\mathrm{d}t}$ acts on $L(g)$;

$$[d_m, g \otimes t^n] = -ng \otimes t^{m+n}. \tag{3}$$

Thus we have the natural semi-direct product $L(g), \oplus\delta$ as a Lie algebra. The affine-Virasoro algebra $\hat{g} \oplus \mathrm{Vir}$ attached to the Lie algebra g is defined as the central extension of $L(g) \oplus \delta$ by $\mathbb{C}c \otimes \mathbb{C}\tilde{c}$. To be more precise, we have the following:

Definition. Let $X_i (i = 1, 2 \ldots, n)$ be a basis of an arbitrary Lie algebra g. Then the affine-Virasoro algebra $\hat{g} \oplus \mathrm{Vir}$ attached to g is defined as a Lie algebra with the basis $\{d_m, X_i(m), c, \tilde{c}; m \in \mathbb{Z},\ i = 1, 2, \ldots, n\}$ and the commutation relations (1), (2) and (3).

Remark. In [5] the commutation relation

$$[d_m, X_i(n)] = 0$$

instead of (3) is introduced in the definition of the affine-Virasoro algebra. It is evident from the context that this is a mistake.

The main purpose of this letter is to construct a bosonic Fock space representation of $\hat{g} \oplus \mathrm{Vir}$ with g being a classical Lie algebra. To this end, we will begin with the construction of such a representation of an affine Lie algebra, then we turn to consider the Virasoro algebra, and finally we combine the obtained results in a natural way.

For convenience, we prepare the following definition of a set I and two functions $Y_\pm$ on $\mathbb{Z}$, the integer ring:

$$I = \{(m_\alpha)_{\alpha\in\mathbb{Z}} | m_\alpha \in \mathbb{Z}^+, m_\alpha = 0 \text{ for } |\alpha| \gg 0\}$$

$$Y_+(\alpha) = \begin{cases} 1 & \text{if } \alpha \geqslant 0 \\ 0 & \text{if } \alpha < 0 \end{cases} \qquad Y_-(\alpha) = 1 - Y_+(\alpha).$$

Then, we consider the boson algebra generated by the infinitely many creation and annihilation operators $a_i^+(\alpha)$, $a_i(\alpha) (i = 1, 2, \ldots, N, \alpha \in \mathbb{Z})$ with the following commutation relations

$$[a_i(\alpha), a_j^+(\beta)] = \delta_{ij}\delta_{\alpha\beta}$$

$$[a_i^+(\alpha), a_j^+(\beta)] = [a_i(\alpha), a_j(\beta)] = 0$$

and the Fock space $\hat{\mathcal{F}}_N(\infty)$ associated with it:

$$\hat{\mathcal{F}}_N(\infty) = \mathrm{Span}\{\Pi_{i,\alpha}\chi_i^{+m_{i\alpha}}(\alpha)|0\rangle | (m_{i\alpha})_\alpha \in I,\ \chi_i^-(\alpha)|0\rangle = 0, i = 1, 2, \ldots, N,\ \alpha \in \mathbb{Z}\}.$$

Here, we have used the notations

$$\chi_i^+(\alpha) = Y_+(\alpha)a_i^+(\alpha) + Y_-(\alpha)a_i(\alpha) \qquad \chi_i^-(\alpha) = Y_-(\alpha)a_i^+(\alpha) + Y_+(\alpha)a_i(\alpha).$$

Let g be an arbitrary Lie algebra and (ρ, V) be an N-dimensional representation of it, where V is the representation space and ρ the homomorphic mapping from g to End$V$. For each $X(m) \in \hat{g}$, consider the operator

$$\pi(X(m)) = \sum_{\alpha\in\mathbb{Z}} \sum_{i,j=1}^{N} (\rho X)_{ij} : a_i^+(\alpha - m)a_j(\alpha) : +\lambda \sum_{i,j=1}^{N} (\rho X)_{ij} a_i^+(-m) \qquad \lambda \in \mathbb{C} \tag{4}$$

on $\hat{\mathcal{F}}_N(\infty)$. Here the normal product : : is defined as

$$: a_i^+(\alpha)a_j(\beta) := a_i^+(\alpha)a_j(\beta) + \delta_{ij}\delta_{\alpha\beta}Y_-(\alpha)$$

i.e. the effect of imposing the normal product is to bring the annihilation operator to the right. As a result, the well-definedness of the operator $\pi(X(m))$ directly follows. Now, the following lemma is easy to verify.

Lemma 1.

$$[\pi(X(m)), a_i^+(\beta)] = \sum_{j=1}^{N} (\rho X)_{ji} a_j^+(\beta - m)$$

$$[\pi(X(m)), a_i(\beta)] = -\sum_{j=1}^{N} (\rho X)_{ij} a_j(\beta + m) - \lambda \sum_{j=1}^{N} (\rho X)_{ij} \delta_{\beta-m}$$

$$[\pi(X(m)), \pi(Y(n))] = \pi([X, Y](m + n)) + m\delta_{m,-n} \mathrm{Tr}(\rho(X)\rho(Y)).$$

We recall that for a classical Lie algebra, the bilinear form $\langle\, ,\, \rangle$ can be chosen as

$$\langle X, Y\rangle = \mathrm{Tr}(\rho(X)\rho(Y))$$

where ρ is the fundamental representation. Thus if we define $\pi(c) = 1$, the following proposition is an immediate consequence of lemma 1.

Proposition 2. Let g be a classical Lie algebra and let ρ be the fundamental representation. Then $(\pi, \hat{\mathcal{F}}_N(\infty))$ is a representation of the affine Lie algebra $\hat{g}$.

We now turn to construct a bosonic Fock representation of the Virasaro algebra. Defining a mapping ϕ from Vir to the operator algebra on $\hat{\mathcal{F}}_1(\infty)$ as

$$\begin{aligned} \phi(d_m) &= \sum_{\alpha\in\mathbb{Z}} \alpha : a_i^+(\alpha - m)a_1(\alpha) :\equiv \sum_{\alpha\in\mathbb{Z}} \alpha : a^+(\alpha - m)a(\alpha) : \\ \phi(\tilde{c}) &= 2 \end{aligned} \tag{5}$$

we can prove the following.

Proposition 3. $(\phi, \hat{\mathcal{F}}_1(\infty))$ is a representation of the Virasoro algebra.

Proof. It is evident that ϕ is well defined. For the commutation relation we have

$$\left[\sum_{\alpha\in\mathbb{Z}}\alpha : a^+(\alpha-m)a(\alpha) :, a^+(\beta)\right] = \beta a^+(\beta-m)$$

$$\left[\sum_{\alpha\in\mathbb{Z}}\alpha : a^+(\alpha-m)a(\alpha) :, a(\beta)\right] = -(\beta+m)a(\beta+m).$$

Thus

$$\begin{aligned}
[\phi(d_m),\phi(d_n)] &= \sum_{\beta\in\mathbb{Z}}\left[\sum_{\alpha\in\mathbb{Z}}\alpha : a^+(\alpha-m)a(\alpha) :, \beta : a^+(\beta-n)a(\beta) :\right] \\
&= \sum_{\beta\in\mathbb{Z}}\beta(a^+(\beta-n)\left[\sum_{\alpha\in\mathbb{Z}}\alpha : a^+(\alpha-m)a(\alpha) :, a(\beta)\right] \\
&\quad + \left[\sum_{\alpha\in\mathbb{Z}}\alpha : a^+(\alpha-m)a(\alpha) :, a^+(\beta-n)\right]a(\beta)) \\
&= \sum_{\beta\in\mathbb{Z}}\beta(-a^+(\beta-n)(\beta+m)a(\beta+m) + (\beta-n)a^+(\beta-m-n)a(\beta)) \\
&= \sum_{\beta\in\mathbb{Z}}(\beta(\beta-n)a^+(\beta-m-n)a(\beta) - \beta(\beta+m)a^+(\beta-n)a(\beta+m)) \\
&= \sum_{\beta\in\mathbb{Z}}((\beta-n)\beta(: a^+(\beta-m-n)a(\beta) : -\delta_{m-n}Y_-(\beta)) \\
&\quad - \beta(\beta+m)(: a^+(\beta-n)a(\beta+m) : -\delta_{m-n}Y_-(\beta-n)) \\
&= \sum_{\beta\in\mathbb{Z}}((\beta-n)\beta - \beta(\beta-m)) : a^+(\beta-m-n)a(\beta) : \\
&\quad - \delta_{m-n}\sum_{\beta\in\mathbb{Z}}((\beta-n)\beta Y_-(\beta) - (\beta-n)\beta Y_-(\beta-n)) \\
&= \sum_{\beta\in\mathbb{Z}}\beta(m-n) : a^+(\beta-m-n)a(\beta) : +\delta_{m-n}(Y_+(n) - Y_-(n))\sum_{\beta=0}^{n}(\beta-n)\beta \\
&= (m-n)\phi(d_{m+n}) + \delta_{m-n}\tfrac{1}{6}(m^3-m).
\end{aligned}$$

Here, in the proof we have used the formula

$$\sum_{\beta=0}^{n}\beta^2 = \tfrac{1}{6}|n|(|n|+1)(1|n|+1).$$

Having obtained Fock representations of the Virasoro algebra and affine Lie algebras, we are prepared to prove our main result. First, let us establish a lemma. We define the following operators on $\hat{\mathcal{F}}_N(\infty)$:

$$\tilde{\phi}(d_m) = \sum_{i=1}^{N}\sum_{\alpha\in\mathbb{Z}}\alpha : a_i^+(\alpha-m)a_i(\alpha) :, \qquad \tilde{\phi}(\tilde{c}) = 2N$$

then it follows directly from proposition 3 that $(\tilde{\phi}, \hat{\mathcal{F}}_N(\infty))$ is also a representation of the Virasoro algebra, and we have:

Lemma 4.

$$[\tilde{\phi}(d_m), \pi(X(n))] = -n\pi(X(m+n)) - \delta_{m-n}\frac{|n|^2+|n|}{2}\mathrm{Tr}(\rho X).$$

Proof. The lemma is proved by calculating the following commutation relations.

$$\left[\tilde{\phi}(d_m), \sum_{k,l=1}^{N}\sum_{\beta\in\mathbb{Z}}(\rho X)_{kl} : a_k^+(\beta-n)a_l(\beta) :\right]$$

$$= \sum_{i,k,l=1}^{N}\sum_{\alpha,\beta,\in\mathbb{Z}}[\alpha : a_i^+(\alpha-m)a_i(\alpha) :, (\rho X)_{kl} : a_k^+(\beta-n)a_l(\beta) :]$$

$$= \sum_{i,k,l=1}^{N}\sum_{\beta\in\mathbb{Z}}\sum_{\alpha\in\mathbb{Z}}\alpha(\rho X)_{kl}(a_i^+(\alpha-m)a_l(\beta)\delta_{i,k}\delta_{\alpha,\beta-n}$$
$$- a_k^+(\beta-n)a_i(\alpha)\delta_{i,l}\delta_{\alpha-m,\beta})$$

$$= \sum_{k,l=1}^{N}\sum_{\beta\in\mathbb{Z}}((\beta-n)(\rho X)_{kl}a_k^+(\beta-m-n)a_l(\beta)$$
$$- (\beta+m)(\rho X)_{kl}a_k^+(\beta-n)a_l(\beta+m))$$

$$= \sum_{k,l=1}^{N}\sum_{\beta\in\mathbb{Z}}((\beta-n)(\rho X)_{kl} : a_k^+(\beta-m-n)a_l(\beta) :$$
$$- (\beta+m)(\rho X)_{kl} : a_k^+(\beta-n)a_l(\beta+m))$$
$$- \sum_{k,l=1}^{N}\sum_{\beta\in\mathbb{Z}}((\beta-n)(\rho X)_{kl}\delta_{k,l}\delta_{m-n}Y_-(\beta) - (\beta+m)(\rho X)_{kl}\delta_{k,l}\delta_{m-n}Y_-(\beta-n))$$

$$= \sum_{k,l=1}^{N}\sum_{\beta\in\mathbb{Z}}((\beta-n)(\rho X)_{kl} : a_k^+(\beta-m-n)a_l(\beta) : -\beta(\rho X)_{kl}$$
$$\times : a_k^+(\beta-m-n)a_l(\beta) :) - \delta_{m,-n}\,\mathrm{Tr}(\rho X)\sum_{\beta\in\mathbb{Z}}(\beta-n)(Y_-(\beta) - Y_-(\beta-n))$$

$$= -n\sum_{k,l=1}^{N}\sum_{\beta\in\mathbb{Z}}(\rho X)_{kl} : a_k^+(\beta-m-n)a_l(\beta) : -\delta_{m,-n}\mathrm{Tr}(\rho X)\frac{|n|^2+|n|}{2}$$

$$\left[\tilde{\phi}(d_m), \sum_{k,l=1}^{N}(\rho X)_{kl}a_k^+(-n)\right] = \sum_{i,k,l=1}^{N}\sum_{\alpha\in\mathbb{Z}}[\alpha : a_i^+(\alpha-m)a_i(\alpha) :, (\rho X)_{kl}a_k^+(-n)]$$

$$= \sum_{i,k,l=1}^{N}\sum_{\alpha\in\mathbb{Z}}(\rho X)_{kl}\alpha a_i^+(\alpha-m)\delta_{i,k}\delta_{\alpha,-n} = -n\sum_{k,l=1}^{N}(\rho X)_{kl}a_k^+(-m-n).$$

A combination gives $[\tilde{\phi}(d_m), \pi(X(n))]$ as the lemma states.

We have proved the lemma for an arbitrary Lie algebra. Now, for our purpose, let us return to the special case where g is a classical Lie algebra and where ρ is its fundamental representation. As is well known, in this case, we have

$$\mathrm{Tr}(\rho X) = 0 \qquad \forall X \in g.$$

Consequently, the following proposition, which is the main result of this letter, comes directly after propositions 2, 3 and lemma 4.

Proposition 5. Let g be a classical Lie algebra and let be its fundamental representation of dimension N. Define the following operators on $\hat{\mathcal{F}}_N(\infty)$:

$$\psi(d_m) = \sum_{\alpha\in\mathbb{Z}} \sum_{i=1}^{N} \alpha : a_i^+(\alpha - m) a_i(\alpha) :, \psi(\tilde{c}) = 2N$$

$$\psi(X(m)) = \sum_{\alpha\in\mathbb{Z}} \sum_{i,j=1}^{N} (\rho X)_{ij} : a_i^+(\alpha - m) a_j(\alpha) : +\lambda \sum_{i,j=1}^{N} (\rho X)_{ij} a_i^+(-m), \lambda \in \mathbb{C}$$

$$\psi(c) = 1.$$

Then $(\psi, \hat{\mathcal{F}}_N(\infty))$ is a representation of the affine-Viarsoro algebra $\hat{g} \oplus \mathrm{Vir}$.

Finally, to conclude this letter, let us give an automorphism φ_z, parameterized by a non-zero integer z, of the affine-Virasoro algebra, by virtue of which one is able to obtain a series of Fock representations of the affine-Virasoro algebra with different centres through the mapping $\psi \cdot \varphi_z$.

Proposition 6. The linear mapping φ_z determined by the equations

$$\varphi_z(d_m) = \frac{1}{z} d_{mz} \qquad \varphi_z(d_0) = \frac{1}{z} d_0 + \frac{\tilde{c}}{24}\left(z - \frac{1}{z}\right)$$

$$\varphi_z(\tilde{c}) = z\tilde{c}$$

$$\varphi_z(X(m)) = X(mz) \qquad \varphi_z(c) = zc$$

is an autormorphism of the affine-Virasoro algebra.

The proof is quite easy, we would rather omit it here.

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